Triple Representation Theorem for orthocomplete homogeneous effect algebras

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Abstract. The aim of our paper is twofold. First, we thoroughly study the set of meager elements $\mathrm{M}(E)$, the set of sharp elements $\mathrm{S}(E)$ and the center $\mathrm{C}(E)$ in the setting of meager-orthocomplete homogeneous effect algebras E. Second, we prove the Triple Representation Theorem for sharply dominating meager-orthocomplete homogeneous effect algebras, in particular orthocomplete homogeneous effect algebras.

Introduction

Two equivalent quantum structures, D-posets and effect algebras were introduced in the nineties of the twentieth century. These were considered as "unsharp" generalizations of the structures which arise in quantum mechanics, in particular, of orthomodular lattices and MV-algebras. Effect algebras aim to describe "unsharp" event structures in quantum mechanics in the language of algebra.

Effect algebras are fundamental in investigations of fuzzy probability theory too. In the fuzzy probability frame, the elements of an effect algebra represent fuzzy events which are used to construct fuzzy random variables.

The aim of our paper is twofold. First, we thoroughly study the set of meager elements $\mathrm{M}(E)$, the set of sharp elements $\mathrm{S}(E)$ and the center $\mathrm{C}(E)$ in the setting of meager-orthocomplete homogeneous effect algebras E. Second, in Section 4 we prove the Triple Representation Theorem, which was established by Jenča in [13] in the setting of complete lattice effect algebras, for sharply

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dominating meager-orthocomplete homogeneous effect algebras, in particular orthocomplete homogeneous effect algebras.

As a by-product of our study we show that an effect algebra E is Archimedean if and only if the corresponding generalized effect algebra $\mathrm{M}(E)$ is Archimedean and that any homogeneous meager-orthocomplete sharply dominating effect algebra E can be covered by Archimedean Heyting effect algebras which form blocks.

1. Preliminaries and basic facts

Effect algebras were introduced by Foulis and Bennett (see [6]) for modelling unsharp measurements in a Hilbert space. In this case the set $\mathcal{E}(H)$ of effects is the set of all self-adjoint operators A on a Hilbert space H between the null operator 0 and the identity operator 1 and endowed with the partial operation $+$ defined iff $A+B$ is in $\mathcal{E}(H)$, where $+$ is the usual operator sum.

In general form, an effect algebra is in fact a partial algebra with one partial binary operation and two unary operations satisfying the following axioms due to Foulis and Bennett.

Definition 1.1. [20] A partial algebra $(E;\oplus,0,1)$ is called an *effect algebra* if 0, 1 are two distinct elements, called the *zero* and the *unit* element, and $\oplus$ is a partially defined binary operation called the *orthosummation* on E which satisfy the following conditions for any $x,y,z\in E$:

(Ei) : $x\oplus y=y\oplus x$ if $x\oplus y$ is defined,
(Eii) : $(x\oplus y)\oplus z=x\oplus(y\oplus z)$ if one side is defined,
(Eiii): for every $x\in E$ there exists a unique $y\in E$ such that $x\oplus y=1$ (we put $x'=y$),
(Eiv) : if $1\oplus x$ is defined then $x=0$.

$(E;\oplus,0,1)$ is called an *orthoalgebra* if $x\oplus x$ exists implies that $x=0$ (see [7]).

We often denote the effect algebra $(E;\oplus,0,1)$ briefly by E. On every effect algebra E a partial order $\leqslant$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x\leqslant y \ \text{ and } \ y\ominus x=z \ \text{ iff } x\oplus z \ \text{ is defined and } x\oplus z=y\,.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E;\oplus,0,1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

Mappings from one effect algebra to another one that preserve units and orthosums are called *morphisms of effect algebras*, and bijective morphisms of effect algebras having inverses that are morphisms of effect algebras are called *isomorphisms of effect algebras*.

Definition 1.2. Let E be an effect algebra. Then $Q\subseteq E$ is called a *sub-effect algebra* of E if

(i) $1\in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

If E is a lattice effect algebra and Q is a sub-lattice and a sub-effect algebra of E, then Q is called a *sub-lattice effect algebra* of E.

Note that a sub-effect algebra Q (sub-lattice effect algebra Q) of an effect algebra E (of a lattice effect algebra E) with inherited operation $\oplus$ is an effect algebra (lattice effect algebra) in its own right.

Definition 1.3. *(1)*: A *generalized effect algebra* $(E;\,\oplus,\,0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying, for any $x, y, z \in E$, conditions

(GE1) $x \oplus y = y \oplus x$ if one side is defined,
(GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,
(GE3) if $x \oplus y = x \oplus z$ then $y = z$,
(GE4) if $x \oplus y = 0$ then $x = y = 0$,
(GE5) $x \oplus 0 = x$ for all $x \in E$.

(2): A binary relation $\leqslant$ (being a partial order) and a partial binary operation $\ominus$ on E can be defined by:

$$x \leqslant y \;\text{ and }\; y \ominus x = z \;\text{ iff } x \oplus z \;\text{ is defined and } x \oplus z = y\,.$$

(3): A nonempty subset $Q \subseteq E$ is called a *sub-generalized effect algebra* of E if out of elements $x, y, z \in E$ with $x \oplus y = z$ at least two are in Q then $x, y, z \in Q$. Then Q is a generalized effect algebra in its own right.

For an element x of a generalized effect algebra E we write $\operatorname{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\operatorname{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. A generalized effect algebra E is *Archimedean* if $\operatorname{ord}(x) < \infty$ for all $x \in E$.

Every effect algebra is a generalized effect algebra.

Definition 1.4. We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of a generalized effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus\limits_{k=1}^{n} x_k$ or $\bigoplus F$) exists in E. Here we define $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ supposing that $\bigoplus\limits_{k=1}^{n-1} x_k$ is defined and $(\bigoplus\limits_{k=1}^{n-1} x_k) \oplus x_n$ exists. We also define $\bigoplus \emptyset = 0$. An arbitrary system $G = (x_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is called *orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a orthogonal system $G = (x_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. We say that $\bigoplus G$ is the *orthogonal sum* of G and G is *orthosummable*. (Here we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (x_\kappa)_{\kappa \in H_1}$). We denote $G^{\oplus} := \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. G is called *bounded* if there is an upper bound of $G^{\oplus}$.

Note that, in any effect algebra E, the following infinite distributive law holds (see [5, Proposition 1.8.7]):

$$\Big(\bigvee_{\alpha} c_\alpha\Big) \oplus b = \bigvee_{\alpha} (c_\alpha \oplus b) \tag{IDL}$$

provided that $\bigvee_\alpha c_\alpha$ and $(\bigvee_\alpha c_\alpha) \oplus b$ exist. We then have the following proposition.

Proposition 1.5. *Let E be a generalized effect algebra, $G_1 = (x_\kappa)_{\kappa\in H_1}$ and $G_2 = (x_\kappa)_{\kappa\in H_2}$ be orthosummable orthogonal systems in E such that $H_1 \cap H_2 = \emptyset$ and $\bigoplus G_1 \oplus \bigoplus G_2$ exists. Then $G = (x_\kappa)_{\kappa\in H_1\cup H_2}$ is an orthosummable orthogonal system and $\bigoplus G = \bigoplus G_1 \oplus \bigoplus G_2$.*

Proof. For any finite $N_1 \subseteq H_1$ and any finite $N_2 \subseteq H_2$, we have that the orthosum $\bigoplus_{\kappa\in N_1} x_\kappa \oplus \bigoplus_{\kappa\in N_2} x_\kappa \leq \bigoplus G_1 \oplus \bigoplus G_2$ exists. Hence G is an orthogonal system.

Evidently, $\bigoplus G_1 \oplus \bigoplus G_2 \geq \bigoplus F$ for any finite system $F = (x_\kappa)_{\kappa\in H}$, $H \subseteq H_1 \cup H_2$ finite. Therefore, $\bigoplus G_1 \oplus \bigoplus G_2$ is an upper bound of $G^{\oplus}$. Let $z \in E$ be any upper bound of $G^{\oplus}$. Then, for any finite $N_1 \subseteq H_1$ and any finite $N_2 \subseteq H_2$, we have that $z \geq \bigoplus_{\kappa\in N_1} x_\kappa \oplus \bigoplus_{\kappa\in N_2} x_\kappa$. Therefore $z \ominus \bigoplus_{\kappa\in N_1} x_\kappa \geq \bigoplus_{\kappa\in N_2} x_\kappa$ and $z \ominus \bigoplus_{\kappa\in N_1} x_\kappa$ is an upper bound of $G_2^{\oplus}$. This yields that, for any finite $N_1 \subseteq H_1$, $z \ominus \bigoplus_{\kappa\in N_1} x_\kappa \geq \bigoplus G_2$, i.e., $z \ominus \bigoplus G_2 \geq \bigoplus_{\kappa\in N_1} x_\kappa$. Hence $z \ominus \bigoplus G_2$ is an upper bound of $G_1^{\oplus}$. Summing up, $z \geq \bigoplus G_1 \oplus \bigoplus G_2$ and this gives that G is orthosummable and $\bigoplus G = \bigoplus G_1 \oplus \bigoplus G_2$. □

Definition 1.6. A generalized effect algebra E is called *orthocomplete* if every bounded orthogonal system is orthosummable.

Observation 1.7. Let E be an orthocomplete generalized effect algebra, $x, y \in E$ such that $ny \leq x$ for every positive integer n. Then $y = 0$.

Proof. Let $G = (g_n)_{n=1}^{\infty}$, $g_n = y$ for every positive integer n. Then, for all $K \subseteq G$ finite, we have $\bigoplus K \leq x$, hence G is bounded and $\bigoplus G$ exists. In virtue of (IDL), $\bigoplus G = g_1 \oplus \bigoplus (g_n)_{n=2}^{\infty} = g_1 \oplus \bigoplus G$. Therefore $0 = g_1 = y$. □

Let us remark a well known fact that every orthocomplete effect algebra is Archimedean.

Definition 1.8. An element x of an effect algebra E is called

(i) *sharp* if $x \wedge x' = 0$. The set $\mathrm{S}(E) = \{x \in E \mid x \wedge x' = 0\}$ is called a *set of all sharp elements* of E (see [10]).

(ii) *principal*, if $y \oplus z \leq x$ for every $y, z \in E$ such that $y, z \leq x$ and $y \oplus z$ exists.

(iii) *central*, if x and x' are principal and, for every $y \in E$ there are $y_1, y_2 \in E$ such that $y_1 \leq x, y_2 \leq x'$, and $y = y_1 \oplus y_2$ (see [9]). The *center* $\mathrm{C}(E)$ of E is the set of all central elements of E.

If $x \in E$ is a principal element, then x is sharp and the interval $[0, x]$ is an effect algebra with the greatest element x and the partial operation given by restriction of $\oplus$ to $[0, x]$.

Statement 1.9. [9, Theorem 5.4] The center $\mathrm{C}(E)$ of an effect algebra E is a sub-effect algebra of E and forms a Boolean algebra. For every central element x of E, $y = (y \wedge x) \oplus (y \wedge x')$ for all $y \in E$. If $x, y \in \mathrm{C}(E)$ are orthogonal, we have $x \vee y = x \oplus y$ and $x \wedge y = 0$.

Statement 1.10. [15, Lemma 3.1.] Let E be an effect algebra, $x, y \in E$ and $c, d \in \mathrm{C}(E)$. Then:

(i) If $x \oplus y$ exists then $c \wedge (x \oplus y) = (c \wedge x) \oplus (c \wedge y)$.
(ii) If $c \oplus d$ exists then $x \wedge (c \oplus d) = (x \wedge c) \oplus (x \wedge d)$.

Definition 1.11. A subset M of a generalized effect algebra E is called *internally compatible* (*compatible*) if for every finite subset M_F of M there is a finite orthogonal family $(x_1, \dots, x_n)$ of elements from M (E) such that for every $m \in M_F$ there is a set $A_F \subseteq \{1, \dots, n\}$ with $m = \bigoplus_{i \in A_F} x_i$. If $\{x, y\}$ is a compatible set, we write $x \leftrightarrow y$ (see [13, 16]).

Evidently, $x \leftrightarrow y$ iff there are $p, q, r \in E$ such that $x = p \oplus q$, $y = q \oplus r$ and $p \oplus q \oplus r$ exists iff there are $c, d \in E$ such that $d \leq x \leq c$, $d \leq y \leq c$ and $c \ominus x = y \ominus d$. Moreover, if $x \wedge y$ exists then $x \leftrightarrow y$ iff $x \oplus (y \ominus (x \wedge y))$ exists.

2. Orthocomplete homogeneous effect algebras

Definition 2.1. An effect algebra E satisfies the *Riesz decomposition property* (or RDP) if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$, there are u_1, u_2 such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.
A lattice effect algebra in which RDP holds is called an *MV-effect algebra*.
An effect algebra E is called *homogeneous* if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$, there are u_1, u_2 such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$ (see [12]).

Lemma 2.2. *Let E be a homogeneous effect algebra and let $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$ and $v_1 \in \mathrm{S}(E)$. Then $u \leq v_2$ and $u \wedge v_1 = 0$.*

Proof. Since E is homogeneous , there are u_1, u_2 such that $u_1 \leq v_1 \leq u'$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$. Let $w \in E$ such that $w \leq u, v_1$. Then $w \leq v_1 \wedge v_1' = 0$. Therefore also $u_1 \leq u \wedge v_1 = 0$, i.e. $u = u_2 \leq v_2$. □

Statement 2.3. [13, Proposition 2]

(i) Every orthoalgebra is homogeneous.
(ii) Every lattice effect algebra is homogeneous.
(iii) An effect algebra E has the Riesz decomposition property if and only if E is homogeneous and compatible.

Let E be a homogeneous effect algebra.

(iv) A subset B of E is a maximal sub-effect algebra of E with the Riesz decomposition property (such B is called a *block* of E) if and only if B is a maximal internally compatible subset of E containing 1.
(v) Every finite compatible subset of E is a subset of some block. This implies that every homogeneous effect algebra is a union of its blocks.
(vi) $\mathrm{S}(E)$ is a sub-effect algebra of E.
(vii) For every block B, $\mathrm{C}(B) = \mathrm{S}(E) \cap B$.
(viii) Let $x \in B$, where B is a block of E. Then $\{y \in E \mid y \leq x \text{ and } y \leq x'\} \subseteq B$.

Hence the class of homogeneous effect algebras includes orthoalgebras, effect algebras satisfying the Riesz decomposition property and lattice effect algebras.

An important class of effect algebras was introduced by Gudder in [10] and [11]. Fundamental example is the standard Hilbert spaces effect algebra $\mathcal{E}(\mathcal{H})$.

For an element x of an effect algebra E we denote

$$\begin{array}{lll} \widetilde{x} & = \bigvee_E \{s \in \mathrm{S}(E) \mid s \leq x\} & \text{if it exists and belongs to } \mathrm{S}(E) \\ \widehat{x} & = \bigwedge_E \{s \in \mathrm{S}(E) \mid s \geq x\} & \text{if it exists and belongs to } \mathrm{S}(E). \end{array}$$

Definition 2.4. ([10], [11].) An effect algebra $(E, \oplus, 0, 1)$ is called *sharply dominating* if for every $x \in E$ there exists $\widehat{x}$, the smallest sharp element such that $x \leq \widehat{x}$. That is $\widehat{x} \in \mathrm{S}(E)$ and if $y \in \mathrm{S}(E)$ satisfies $x \leq y$ then $\widehat{x} \leq y$.

Recall that evidently an effect algebra E is sharply dominating iff for every $x \in E$ there exists $\widetilde{x} \in \mathrm{S}(E)$ such that $\widetilde{x} \leq x$ and if $u \in \mathrm{S}(E)$ satisfies $u \leq x$ then $u \leq \widetilde{x}$ iff for every $x \in E$ there exist a smallest sharp element $\widehat{x}$ over x and a greatest sharp element $\widetilde{x}$ below x.

In what follows set (see [13, 21])

$$\mathrm{M}(E) = \{x \in E \mid \text{ if } v \in \mathrm{S}(E) \text{ satisfies } v \leq x \text{ then } v = 0\}.$$

An element $x \in \mathrm{M}(E)$ is called *meager*. Moreover, $x \in \mathrm{M}(E)$ iff $\widetilde{x} = 0$. Recall that $x \in \mathrm{M}(E)$, $y \in E$, $y \leq x$ implies $y \in \mathrm{M}(E)$ and $x \ominus y \in \mathrm{M}(E)$.

Definition 2.5. Let E be an effect algebra and let

$$\mathrm{HM}(E) = \{x \in E \mid \text{ there is } y \in E \text{ such that } x \leq y \text{ and } x \leq y'\}.$$

An element $x \in \mathrm{HM}(E)$ is called *hypermeager*.

Every hypermeager element is meager. Since both $\mathrm{M}(E)$ and $\mathrm{HM}(E)$ are downsets of E they form together with the corresponding restriction of the operation $\oplus$ a generalized effect algebra.

Lemma 2.6. *(1) Let* $\mathrm{ord}(y) = \infty$ *in* E. $\{ky \mid k \in \mathbb{N}\} \subseteq \mathrm{HM}(E)$.
(2)Let $\mathrm{ord}(y) = n_y \neq \infty$ *in* E. $\{ky \mid k \in \mathbb{N}, k \leq \frac{n_y}{2}\} \subseteq \mathrm{HM}(E)$.

Proof. In either case, $(2k)y$ exists in E. Therefore $ky \leq (ky)'$ and consequently $ky \in \mathrm{HM}(E)$. □

Proposition 2.7. *Let E be an effect algebra. The following conditions are equivalent.*

(i) *E is Archimedean;*
(ii) $\mathrm{M}(E)$ *is Archimedean;*
(iii) $\mathrm{HM}(E)$ *is Archimedean.*

Proof. (i) $\Longrightarrow$ (ii) If E is Archimedean, $\mathrm{M}(E)$ is Archimedean a fortiori.
(ii) $\Longrightarrow$ (iii) If $\mathrm{M}(E)$ is Archimedean, $\mathrm{HM}(E)$ is Archimedean a fortiori.
(iii) $\Longrightarrow$ (i) Let $\mathrm{HM}(E)$ be Archimedean. Suppose $\mathrm{ord}(y) = \infty$ in E where $y \neq 0$. By Lemma 2.6, $\{ky \mid k \in \mathbb{N}\} \subseteq \mathrm{HM}(E)$, which contradicts the assumption. $\square$

Statement 2.8. [17, Lemma 2.4] Let E be an effect algebra in which $\mathrm{S}(E)$ is a sub-effect algebra of E and let $x \in \mathrm{M}(E)$ such that $\widehat{x}$ exists. Then

(i) $\widehat{x} \ominus x \in \mathrm{M}(E)$.
(ii) If $y \in \mathrm{M}(E)$ such that $x \oplus y$ exists and $x \oplus y = z \in \mathrm{S}(E)$ then $\widehat{x} = z$.

Statement 2.9. [17, Lemma 2.5] Let E be an effect algebra in which $\mathrm{S}(E)$ is a sub-effect algebra of E and let $x \in E$ such that $\widetilde{x}$ exists. Then $x \ominus \widetilde{x} \in \mathrm{M}(E)$ and $x = \widetilde{x} \oplus (x \ominus \widetilde{x})$ is the unique decomposition $x = x_S \oplus x_M$, where $x_S \in \mathrm{S}(E)$ and $x_M \in \mathrm{M}(E)$. Moreover, $x_S \wedge x_M = 0$ and if E is a lattice effect algebra then $x = x_S \vee x_M$.

As proved in [1], $\mathrm{S}(E)$ is always a sub-effect algebra in a sharply dominating effect algebra E.

Corollary 2.10. [13, Proposition 15] *Let E be a sharply dominating effect algebra. Then every $x \in E$ has a unique decomposition $x = x_S \oplus x_M$, where $x_S \in \mathrm{S}(E)$ and $x_M \in \mathrm{M}(E)$, namely $x = \widetilde{x} \oplus (x \ominus \widetilde{x})$.*

Statement 2.11. [13, Corollary 14] Let E be an orthocomplete homogeneous effect algebra. Then E is sharply dominating.

Proposition 2.12. *Let E be a homogeneous effect algebra and $v \in E$. The following conditions are equivalent.*

(i) $v \in \mathrm{S}(E)$*;*
(ii) $y \leq z$ *whenever* $w, y, z \in E$ *such that* $v = w \oplus z$, $y \leq w'$ *and* $y \leq w$.
(iii) $[0, w] \cap [0, w'] = [0, w] \cap [0, v \ominus w]$ *whenever* $w \in E$ *and* $w \leq v$.

Proof. (i) $\Longrightarrow$ (ii): Evidently, there is a block, say B, such that it contains the following orthogonal system $\{y, w \ominus y, z, 1 \ominus v\}$. Hence B contains also w, w' and $v \in \mathrm{C}(B)$. Since $1 = w \oplus w'$ we obtain by Statement 1.10, (ii) that $v = v \wedge_B w \oplus v \wedge_B w' = w \oplus v \wedge_B w'$. Subtracting w we obtain $z = v \wedge_B w'$. Hence $y \leq w \leq v$ and $y \leq w'$ yields that $y \leq z$.
(ii) $\Longrightarrow$ (iii): Clearly, $[0, w] \cap [0, v \ominus w] \subseteq [0, w] \cap [0, w']$. The other inclusion is a direct reformulation of (ii).

(iii) $\implies$ (i) Let $y \in [0, v] \cap [0, v']$. Then from (iii) we have that $y \in [0, v] \cap [0, v \ominus v] = \{0\}$. Immediately, $y = 0$ and v is sharp. □

Corollary 2.13. [18, Lemma 2.12] *Let E be a homogeneous effect algebra, and $y \in E$ and $w \in \mathrm{S}(E)$ for which $y \leq w$ and ky exists. It holds $ky \leq w$.*

Corollary 2.14. *Let E be a homogeneous effect algebra and let $x, y \in E$ such that $\widehat{x}$ exists, $y \leq (\widehat{x} \ominus x)'$ and $y \leq \widehat{x} \ominus x$. Then $y \leq x$.*

Proof. It is enough to put in Proposition 2.12 $v = \widehat{x}$, $w = \widehat{x} \ominus x$ and $z = x$. □

Statement 2.15. [18, Lemma 1.16.] Let E be a sharply dominating effect algebra and let $x \in E$. Then

$$\widehat{x \ominus \widetilde{x}} = \widehat{\widehat{x} \ominus x} = \widehat{x} \ominus \widetilde{x}.$$

Lemma 2.16. *Let E be an effect algebra and let $x \in E$. Then*

(i) *If $\widehat{x}$ exists then $\widetilde{(x')}$ exists and*

$$\widehat{x} \ominus x = x' \ominus (\widehat{x})' = x' \ominus \widetilde{(x')}.$$

(ii) *If $\widetilde{x}$ exists then $\widehat{(x')}$ exists and*

$$x \ominus \widetilde{x} = (\widetilde{x})' \ominus x' = \widehat{(x')} \ominus x'.$$

Proof. Transparent. □

Lemma 2.17. *Let E be an effect algebra and let $x, y \in E$.*

(i) *If $\widehat{x}$ exists then*

$$y \leq \widehat{x} \ominus x \quad \textit{if and only if} \quad y \leq x' \quad \textit{and} \quad \widehat{x \oplus y} = \widehat{x}.$$

(ii) *If $\widetilde{x}$ exists then*

$$y \leq x \ominus \widetilde{x} \quad \textit{if and only if} \quad y \leq x \quad \textit{and} \quad \widetilde{x \ominus y} = \widetilde{x}.$$

Proof. Transparent. □

Lemma 2.18. *Let E be a homogeneous effect algebra and let $x \in E$.*

(i) *If $\widetilde{x}$ exists then*

$$[0, x] \cap [0, x'] = [0, x \ominus \widetilde{x}] \cap [0, x'] = [0, x \ominus \widetilde{x}] \cap [0, (x \ominus \widetilde{x})'].$$

(ii) *If $\widehat{x}$ exists then*

$$[0, x] \cap [0, x'] = [0, x] \cap [0, \widehat{x} \ominus x] = [0, (\widehat{x} \ominus x)'] \cap [0, \widehat{x} \ominus x].$$

(iii) *If both $\widetilde{x}$ and $\widehat{x}$ exist then*

$$[0, x] \cap [0, x'] = [0, x \ominus \widetilde{x}] \cap [0, \widehat{x} \ominus x].$$

Proof. (i): Clearly, $[0,x]\cap[0,x'] \supseteq [0,x\ominus\widetilde{x}]\cap[0,x']$ and $[0,x\ominus\widetilde{x}]\cap[0,x'] \subseteq [0,x\ominus\widetilde{x}]\cap[0,x'\oplus\widetilde{x}] = [0,x\ominus\widetilde{x}]\cap[0,(x\ominus\widetilde{x})']$. Assume that $y\in[0,x]\cap[0,x']$. Then by Proposition 2.12 applied to $\widehat{x'}$ and x' we obtain that $y \le \widehat{x'}\ominus x' = x\ominus\widetilde{x}$. Hence $[0,x]\cap[0,x'] = [0,x\ominus\widetilde{x}]\cap[0,x']$.

Now, assume that $z\in[0,x\ominus\widetilde{x}]\cap[0,(x\ominus\widetilde{x})']$. Then $z\le(x\ominus\widetilde{x})' = x'\oplus\widetilde{x}\le z'$. Therefore $z = z_1\oplus z_2$, $z_1\le x'$ and $z_2\le\widetilde{x}\le z'\le z_2'$. Hence $z_2\le\widetilde{x}\wedge\widetilde{x}' = 0$. This yields that $z = z_1\le x'$, i.e. $[0,x\ominus\widetilde{x}]\cap[0,x'] = [0,x\ominus\widetilde{x}]\cap[0,(x\ominus\widetilde{x})']$.

(ii): It follows by interchanging x and x'.

(iii): We have

$$[0,x]\cap[0,x'] = [0,x\ominus\widetilde{x}]\cap[0,(x\ominus\widetilde{x})']\cap[0,(\widehat{x}\ominus x)']\cap[0,\widehat{x}\ominus x].$$

Since $x\ominus\widetilde{x}\le(\widehat{x}\ominus x)'$ and $\widehat{x}\ominus x\le(x\ominus\widetilde{x})'$ we are finished. □

Lemma 2.19. *Let E be a homogeneous effect algebra and let $x, x_1, \dots, x_n\in E$ be such that $\bigoplus_{i=1}^n x_i \le x$, $\widetilde{x}$ exists and, for all $i = 1,\dots,n$, $x\le x_i'$. Then $\bigoplus_{i=1}^n x_i \le x\ominus\widetilde{x}$ and therefore $\widetilde{x} = \widetilde{x\ominus\bigoplus_{i=1}^n x_i}$.*

Proof. For $n=0$, the statement trivially holds. Assume that the statement is satisfied for some n. Then $\bigoplus_{i=1}^{n+1} x_i \le x = \widetilde{x}\oplus(x\ominus\widetilde{x})$ and, for all $i = 1,\dots,n{+}1$, $x_i\le x'$ yield that $\bigoplus_{i=1}^n x_i\le x\ominus\widetilde{x}$, and clearly $x_{n+1}\le x\ominus\bigoplus_{i=1}^n x_i$. Since also $x_{n+1}\le x'\le(x\ominus\bigoplus_{i=1}^n x_i)'$, further $x_{n+1}\le\widetilde{x}\oplus((x\ominus\widetilde{x})\ominus\bigoplus_{i=1}^n x_i)\le x_{n+1}'$. By Lemma 2.2 we obtain $x_{n+1}\le(x\ominus\widetilde{x})\ominus\bigoplus_{i=1}^n x_i$. This yields that $\bigoplus_{i=1}^{n+1} x_i\le x\ominus\widetilde{x}$. □

3. Blocks and orthogonal sums of hypermeager elements in meager-orthocomplete effect algebras

Definition 3.1. An effect algebra E is *meager-orthocomplete* if $\mathrm{M}(E)$ is an orthocomplete generalized effect algebra. For a bounded orthogonal family $(v_i)_{i\in I}$ in $\mathrm{M}(E)$ we shall denote by $\bigoplus_{i\in I}^{\mathrm{M}(E)} v_i$ the orthogonal sum of $(v_i)_{i\in I}$ calculated in $\mathrm{M}(E)$.

Observation 3.2. Every orthocomplete effect algebra is meager-orthocomplete and sharply dominating.

Proposition 3.3. *Let E be a homogeneous meager-orthocomplete effect algebra. Let $(v_i)_{i\in I}$ be an orthogonal family such that $v = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i$ exists, $v\in\mathrm{M}(E)$ and $u\in E$ be such that $u\le v\le u'$. Then there is an orthogonal family $(u_i)_{i\in I}$ such that $u = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i$ exists and $u_i\le v_i$ for all $i\in I$.*

Proof. Let us put $X = \{(x,i)\in E\times I \mid i\in I, x\le v_i, x\le u\}$. We say that a subset $Y\subseteq X$ is *u-good* if

(i) For all $f,g\in Y$, $\pi_2(f) = \pi_2(g)$ implies $f = g$.
(ii) $\bigoplus_{y\in Y}^{\mathrm{M}(E)} \pi_1(y)\le u$.
(iii) $u\ominus\bigoplus_{y\in Y}^{\mathrm{M}(E)} \pi_1(y)\le v\ominus\bigoplus_{y\in Y}^{\mathrm{M}(E)} v_{\pi_2(y)}$.

Let us denote $u - X$ the system of all u-good subsets of X ordered by inclusion. Then $\emptyset \in u - X$ and the union of any chain in $u - X$ is again in $u - X$ in virtue of (IDL). Hence by Zorn's lemma there is a maximal element, say Z, in $u - X$.

Let us show that $\pi_2(Z) = I$. Assume the contrary. Then there is $j \in I$ such that $j \notin \pi_2(Z)$. Then

$$\begin{aligned} u \ominus \bigoplus\nolimits_{y\in Z}^{\mathrm{M}(E)} \pi_1(y) \leq v \ominus \bigoplus\nolimits_{y\in Z}^{\mathrm{M}(E)} v_{\pi_2(y)} &= v_j \oplus ((v \ominus \bigoplus\nolimits_{y\in Z}^{\mathrm{M}(E)} v_{\pi_2(y)}) \ominus v_j) \\ &\leq u' \leq (u \ominus \bigoplus\nolimits_{y\in Z}^{\mathrm{M}(E)} \pi_1(y))'. \end{aligned}$$

Since E is homogeneous we get that $u \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y) = u_j \oplus x$ such that $u_j \leq v_j$ and $x \leq (v \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} v_{\pi_2(y)}) \ominus v_j$. Hence $(u \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y)) \ominus u_j = x \leq (v \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} v_{\pi_2(y)}) \ominus v_j$. This yields that the set $Z \cup \{(u_j, j)\}$ is u-good, a contradiction with the maximality of Z.

Therefore $\pi_2(Z) = I$. But this yields $u \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y) \leq v \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} v_{\pi_2(y)} = 0$. Let us put $u_{\pi_2(y)} = \pi_1(y)$ for all $y \in Z$. Hence $u \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i = 0$, i.e., $u = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i$. □

Proposition 3.4. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Let $v_1, v_2 \in E$ such that $v_1 \leq v_2'$. Let $(u_i)_{i\in I}$ be an orthogonal family such that $u = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i \in \mathrm{M}(E)$ exists, $u \leq v_1 \oplus v_2$ and, for all $i \in I$, $v_1 \oplus v_2 \leq u_i'$. Then there are orthogonal families $(v_i^1)_{i\in I}$ and $(v_i^2)_{i\in I}$ such that $w_1 = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^1 \leq v_1$ exists, $w_2 = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^2 \leq v_2$ exists, $u = w_1 \oplus w_2$ and, for all $i \in I$, $v_i^1 \oplus v_i^2 = u_i$, $v_1 \leq {v_i^1}'$ and $v_2 \leq {v_i^2}'$.*

Proof. Let us put $X_{1,2} = \{(x_1, x_2, i) \in E^2 \times I \mid i \in I, x_1 \leq v_1, x_2 \leq v_2\}$. We say that a subset $Y \subseteq X_{1,2}$ is $u_{1,2}$-*good* if

(i) For all $f, g \in Y$, $\pi_3(f) = \pi_3(g)$ implies $f = g$,
(ii) $\pi_1(y) \oplus \pi_2(y) = u_{\pi_3(y)}$ for all $y \in Y$,
(iii) $\bigoplus_{y\in Y}^{\mathrm{M}(E)} \pi_1(y) \leq v_1$,
(iv) $\bigoplus_{y\in Y}^{\mathrm{M}(E)} \pi_2(y) \leq v_2$.

Let us denote $u - X_{1,2}$ the system of all $u_{1,2}$-good subsets of $X_{1,2}$ ordered by inclusion. Then $\emptyset \in u - X_{1,2}$ and the union $Y = \bigcup\{Y_\alpha, \alpha \in \Lambda\}$ of a chain of $u_{1,2}$-good sets $Y_\alpha, \alpha \in \Lambda$ in $u - X_{1,2}$ is again in $u - X_{1,2}$. Namely, the conditions (i) and (ii) are obviously satisfied. Let us check the condition (iii). Let $F \subseteq Y$ be a finite subset of Y. Then there is $\alpha_0 \in \Lambda$ such that $F \subseteq Y_{\alpha_0}$. Hence $\bigoplus_{y\in F} \pi_1(y) = \bigoplus_{y\in F}^{\mathrm{M}(E)} \pi_1(y) \leq v_1 \leq v_1 \oplus v_2 \leq u'_{\pi_3(y)} \leq \pi_1(y)'$ for all $y \in F$. By Lemma 2.19 we get that $\bigoplus_{y\in F} \pi_1(y) \leq v_1 \ominus \widetilde{v_1} \in \mathrm{M}(E)$. Since $\mathrm{M}(E)$ is an orthocomplete generalized effect algebra we have that $\bigoplus_{y\in Y}^{\mathrm{M}(E)} \pi_1(y)$ exists in $\mathrm{M}(E)$ and therefore $\bigoplus_{y\in Y}^{\mathrm{M}(E)} \pi_1(y) \leq v_1 \ominus \widetilde{v_1} \leq v_1$. The condition (iv) follows by similar considerations.

By Zorn's lemma there is a maximal element, say Z, in $u - X_{1,2}$.

Let us show that $\pi_3(Z) = I$. Assume the contrary. Then there is $j \in I$ such that $j \notin \pi_3(Z)$. Therefore by a successive application of Proposition 1.5

$$\begin{aligned} u &= \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i = \bigoplus_{y\in Z}^{\mathrm{M}(E)} u_{\pi_3(y)} \oplus \bigoplus_{k\in I\setminus\pi_3(Z)}^{\mathrm{M}(E)} u_k \\ &= \bigoplus_{y\in Z}^{\mathrm{M}(E)} (\pi_1(y) \oplus \pi_2(y)) \oplus \bigoplus_{k\in I\setminus\pi_3(Z)}^{\mathrm{M}(E)} u_k \\ &= \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y) \oplus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_2(y) \oplus \bigoplus_{k\in I\setminus\pi_3(Z)}^{\mathrm{M}(E)} u_k \leq v_1 \oplus v_2. \end{aligned}$$

The last inequality yields

$$u_j \leq (v_1 \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y)) \oplus (v_2 \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_2(y)) \leq v_1 \oplus v_2 \leq u_j'.$$

Since E is homogeneous we get that there are $v_j^1, v_j^2 \in E$ such that $v_j^1 \oplus v_j^2 = u_j$, $v_j^1 \leq v_1 \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y)$, $v_j^2 \leq v_2 \ominus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_2(y)$.

This yields that the set $Z \cup \{(v_j^1, v_j^2, j)\}$ is $u_{1,2}$-good, a contradiction with the maximality of Z.

Therefore $\pi_3(Z) = I$. For any $i \in I$ there is a unique $y_i \in Z$ such that $\pi_3(y_i) = i$ and, conversely, for any $y \in Z$ there is a unique $i_y \in I$ such that $\pi_3(y) = i_y$. Let us put $v_i^1 = \pi_1(y_i)$, $v_i^2 = \pi_2(y_i)$, $w_1 = \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y)$ and $w_2 = \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_2(y)$. Since Z is $u_{1,2}$-good we have that $w_1 \leq v_1$, $w_2 \leq v_2$ and $w_1 \oplus w_2 = \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_1(y) \oplus \bigoplus_{y\in Z}^{\mathrm{M}(E)} \pi_2(y) = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i$.

Moreover, for all $i \in I$, $v_1 \leq v_1 \oplus v_2 \leq u_i' \leq {v_i^1}'$ and $v_2 \leq v_1 \oplus v_2 \leq u_i' \leq {v_2^1}'$. □

Corollary 3.5. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Let $v_1, \dots, v_k \in E$ be an orthogonal family. Let $(u_i)_{i\in I}$ be an orthogonal family such that $u = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i \in \mathrm{M}(E)$ exists, $u \leq v_1 \oplus \cdots \oplus v_k$ and, for all $i \in I$, $v_1 \oplus \cdots \oplus v_k \leq u_i'$. Then there are orthogonal families $(v_i^1)_{i\in I}$, $\dots$, $(v_i^k)_{i\in I}$ such that $w_1 = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^1 \leq v_1$ exists, $\dots$, $w_k = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^k \leq v_2$ exists, $w_1 \leq v_1$, $\dots$, $w_k \leq v_k$, $u = w_1 \oplus \cdots \oplus w_k$ and, for all $i \in I$, $v_i^1 \oplus \cdots \oplus v_i^k = u_i$, $v_1 \leq {v_i^1}'$, $\dots$, $v_k \leq {v_i^k}'$.*

Proof. Straightforward induction with respect to k. □

Corollary 3.6. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Let $v, u \in \mathrm{M}(E)$, $v \leq u$. Let $(u_i)_{i\in I}$ be an orthogonal family such that $u = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i \in \mathrm{M}(E)$ exists and, for all $i \in I$, $v \oplus (u \ominus v) = u \leq u_i'$. Then there are orthogonal families $(v_i^1)_{i\in I}$, $(v_i^2)_{i\in I}$ such that $v = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^1$ exists, $u \ominus v = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^2$ exists and, for all $i \in I$, $v_i^1 \oplus v_i^2 = u_i$, $v \leq {v_i^1}'$ and $u \ominus v \leq {v_i^k}'$.*

Proof. From Proposition 3.4 we know that there are orthogonal families $(v_i^1)_{i\in I}$ and $(v_i^2)_{i\in I}$ such that $w_1 = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^1 \leq v$ exists, $w_2 = \bigoplus_{i\in I}^{\mathrm{M}(E)} v_i^2 \leq u \ominus v$ exists, $u = w_1 \oplus w_2$ and, for all $i \in I$, $v_i^1 \oplus v_i^2 = u_i$, $v \leq {v_i^1}'$ and $u \ominus v \leq {v_i^2}'$. We have $u = v \oplus (u \ominus v) = w_1 \oplus (v \ominus w_1) \oplus w_2 \oplus ((u \ominus v) \ominus w_2) = u \oplus (v \ominus$

$w_1) \oplus ((u \ominus v) \ominus w_2)$. Therefore $0 = v \ominus w_1 = (u \ominus v) \ominus w_2$. This yields $v = w_1$ and $u \ominus v = w_2$. □

Let E be an meager-orthocomplete effect algebra. Let $u \in E$. We put $\vartheta(u) = \{w \in E \mid w = v \text{ or } w = u \ominus v, (u_i)_{i\in I}$ is an orthogonal family such that $v = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i \in \mathrm{M}(E)$ exists, $v \leq u$ and, for all $i \in I$, $u \leq u_i'\}$. Clearly, $\{0, u\} \subseteq \vartheta(u)$. Recall that, for sharply dominating and homogeneous E and any element $v \in \vartheta(u)$ of the above form $\bigoplus_{i\in I}^{\mathrm{M}(E)} u_i$, we have by Corollary 3.6 $[0, v] \cup [u \ominus v, u] \subseteq \vartheta(u)$. Note also that, for $u \in \mathrm{S}(E)$, we obtain that $\vartheta(u) = \{0, u\}$. Therefore, we have a map $\Theta : 2^E \to 2^E$ defined by $\Theta(A) = \bigcup\{\vartheta(u) \mid u \in A\}$ for all $A \subseteq E$. The above considerations yield that $A \cup \{0\} \subseteq \Theta(A)$.

As in [13, Theorem 8], for any set $A \subseteq E$, $\sigma(A)$ is the smallest superset of A closed with respect to Θ. Clearly, $\sigma(A) = \bigcup_{n=0}^{\infty} A_n$, where A_n are subsets of E given by the rules $A_0 = A$, $A_{n+1} = \Theta(A_n)$.

Lemma 3.7. *Let E be a homogeneous meager-orthocomplete effect algebra. Let $A \subseteq E$ be an internally compatible subset of E. Then $\sigma(A)$ is internally compatible.*

Proof. The proof goes literally the same way as in [13, Theorem 8]. Hence we omit it. □

Corollary 3.8. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. For every block B of E, $\sigma(B) = \Theta(B) = B$.*

Proposition 3.9. *Let E be a homogeneous meager-orthocomplete effect algebra, let $x \in \mathrm{M}(E)$. Let $(x_i)_{i\in I}$ be a maximal orthogonal family such that, for all $i \in I$, $x_i \leq x'$ and, for all $F \subseteq I$ finite, $\bigoplus_{i\in F} x_i \leq x$. Then $x = \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i$.*

Proof. Since $\mathrm{M}(E)$ is an orthocomplete generalized effect algebra we have that $\bigoplus_{i\in I}^{\mathrm{M}(E)} x_i$ exists. As in [13, Theorem 13] we will prove that $x \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \in \mathrm{S}(E)$. Let $r \in E$ be such that

$$r \leq x \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i, (x \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i)'.$$

We get that

$$r \leq (x \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i)' = x' \oplus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \leq r'.$$

Therefore there are $r_1, r_2 \in E$ such that $r = r_1 \oplus r_2$, $r_1 \leq x'$, $r_2 \leq \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i$. We have also that $r_1 \oplus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \leq x$, i.e. $r_1 = 0$ and $r_2 = r$ by maximality of $(x_i)_{i\in I}$. Then $r \leq \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \leq r'$. Proposition 3.3 yields that there is an orthogonal family $(u_i)_{i\in I}$ such that $r = \bigoplus_{i\in I}^{\mathrm{M}(E)} u_i$ exists and $u_i \leq x_i$ for all $i \in I$. Hence, for all $j \in I$,

$$u_j \leq r \leq x \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i, \text{ i.e., } u_j \oplus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \leq x \text{ and } u_j \leq x_j \leq x'.$$

Thus again by maximality of $(x_i)_{i\in I}$ we get that $u_j = 0$. It follows that $r = 0$ and $x \ominus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \in \mathrm{S}(E) \cap \mathrm{M}(E) = \{0\}$, i.e. $x = \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i$. $\square$

Proposition 3.10. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra and let $x \in E$. Let $(x_i)_{i\in I}$ be a maximal orthogonal family such that, for all $i \in I$, $x_i \leq x'$ and, for all $F \subseteq I$ finite, $\bigoplus_{i\in F} x_i \leq x$. Then $x = \widetilde{x} \oplus \bigoplus_{i\in I}^{\mathrm{M}(E)} x_i$. Moreover, for every block B of E, $x \in B$ implies that $[\widetilde{x}, x] \subseteq B$ and $[x, \widehat{x}] \subseteq B$.*

Proof. Note first that by Lemma 2.18 and Lemma 2.19 $(x_i)_{i\in I}$ is a maximal orthogonal family such that, for all $i \in I$, $x_i \leq (x \ominus \widetilde{x})'$ and, for all $F \subseteq I$ finite, $\bigoplus_{i\in F} x_i \leq x \ominus \widetilde{x}$. From Proposition 3.9 we get that $\bigoplus_{i\in I}^{\mathrm{M}(E)} x_i$ exists and $\bigoplus_{i\in I}^{\mathrm{M}(E)} x_i = x \ominus \widetilde{x}$. Let B be a block of E, $x \in B$. Since $\bigoplus_{i\in I}^{\mathrm{M}(E)} x_i \in \vartheta(x) \subseteq B$ we get by Corollary 3.6 that $[0, x \ominus \widetilde{x}] \subseteq B$. Therefore $\widetilde{x} \in B$ and $[\widetilde{x}, x] = \widetilde{x} \oplus [0, x \ominus \widetilde{x}] \subseteq B$. Following the same reasonings for x' we get that $[\widetilde{x'}, x'] \subseteq B$. Hence also $[x, \widehat{x}] = [x, \widetilde{x'}'] \subseteq B$. $\square$

Corollary 3.11. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra, let $x \in \mathrm{M}(E)$. Then, for every block B of E, $x \in B$ implies that $[0, x] \subseteq B$ and, moreover, $\mathrm{M}(B) \subseteq \mathrm{M}(E)$.*

Proof. Since $x \in \mathrm{M}(E)$ we get by Corollary 3.6 and Proposition 3.9 that $[0, x] \subseteq \vartheta(x) \subseteq B$.

Now, let $y \in \mathrm{M}(B) \subseteq B$, $z \in \mathrm{S}(E)$, $z \leq y$. Then $z \leq \widetilde{y} \in B$ and $\widetilde{y} \in \mathrm{S}(B)$. Hence $\widetilde{y} = 0$. This yields $z = 0$. $\square$

As in [13, Proposition 16] we have that

Proposition 3.12. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra, let $x \in \mathrm{M}(E)$. Then $[0, x]$ is a complete MV-effect algebra.*

Proof. Let B be a block containing x. Since $[0, x] \subseteq B$ by Corollary 3.11 and $[0, x]$ is an orthocomplete effect algebra we obtain that $[0, x]$ is an orthocomplete effect algebra satisfying the Riesz decomposition property. From [14, Theorem 4.10] we get that $[0, x]$ is a lattice and hence a complete MV-effect algebra. $\square$

Recall that Proposition 3.12 immediately yields (using the same considerations as in [13, Proposition 19]) that an orthocomplete generalized effect algebra of meager elements of a sharply dominating homogeneous effect algebra is a commutative BCK-algebra with the relative cancellation property. Hence, by the result of J. Cīrulis (see [4]) it is the dual of a weak implication algebra introduced in [2].

Proposition 3.13. *Let E be a meager-orthocomplete effect algebra, and let $y, z \in \mathrm{M}(E)$. Every lower bound of y, z is below a maximal one.*

Proof. Let w be a lower bound of y, z. There exists a maximal orthogonal multiset A containing w in which 0 occurs uniquely and for which y, z are upper bounds of $A^{\oplus}$. Indeed, the multiset union of any maximal chain of such multisets is again such multiset.

Since $\mathrm{M}(E)$ is orthocomplete any non-zero element of A has finite multiplicity. Again by orthocompleteness, there exists a smallest upper bound u of $A^{\oplus}$ below y, z and hence a lower bound of y, z above w. Let v be an arbitrary lower bound of y, z above u. If $u < v$, the multiset sum $A \uplus \{v \ominus u\}$ is an orthogonal multiset satisfying all requirements which is properly larger than A. Hence $u = v$ is a maximal lower bound of y, z over w. □

The proof of the following Proposition follows the proof from [13, Proposition 17].

Proposition 3.14. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Then $\mathrm{M}(E)$ is a meet semilattice.*

Proof. Let $x, y \in \mathrm{M}(E)$. From Proposition 3.13, every lower bound of x, y is under a maximal lower bound of x, y. Let u, v be maximal lower bounds of x, y. By Proposition 3.12, $[0, x]$ and $[0, y]$ are complete MV-effect algebras. Denote by $z = u \wedge_{[0,x]} v \in [0, y]$. Then $z \leq u \wedge_{[0,y]} v$. By a symmetric argument we get that $u \wedge_{[0,y]} v \leq z$, i.e. $z = u \wedge_{[0,y]} v$. We may assume that $z = 0$ otherwise we could shift x, y, u, v, z by z. Since $u \leftrightarrow_{[0,x]} v$ and $u \leftrightarrow_{[0,y]} v$ we get that $u \oplus v \leq x$ exists and $u \oplus v = u \vee_{[0,x]} v = u \vee_{[0,y]} v \leq x, y$, $u \leq u \oplus v$, $v \leq u \oplus v$. Therefore $u = v = 0$, i.e., any two maximal lower bounds of x, y coincide. □

In what follows we will extend and modify [13, Lemma 20, Proposition 21] for an orthocomplete homogeneous effect algebra E.

Proposition 3.15. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra and let $x, y \in E$ are in the same block B of E such that $x \wedge_B y = 0$. Then $\widehat{x} \wedge \widehat{y} = 0$.*

Proof. Note that $\widehat{x}, \widehat{x} \ominus x, \widehat{y}, \widehat{y} \ominus y \in B$. Let us first check that $\widehat{x} \wedge_B y = 0$. By Proposition 3.10 applied to the element $\widehat{x} \ominus x$ we have from Lemma 2.18 that there is an orthogonal family $(x_j)_{j \in J}$ such that, for all $j \in J$, $x_j \leq x, x_j \in B \cap [0, \widehat{x} \ominus x]$ and $\bigoplus_{j \in J}^{\mathrm{M}(E)} x_j = \widehat{x} \ominus x \in \mathrm{M}(E)$. Let $w \in B$, $w \leq \widehat{x}$ and $w \leq y$. Since $w \leq x \oplus (\widehat{x} \ominus x)$ we can find by the Riesz decomposition property of B elements $w_1, w_2 \in B$ such that $w = w_1 \oplus w_2$, $w_1 \leq x$ and $w_2 \leq \widehat{x} \ominus x$. Therefore $w_1 \leq x \wedge y = 0$ implies $w = w_2 \leq \widehat{x} \ominus x$. Hence by Corollary 3.6 we obtain that there exists an orthogonal family $(u_j)_{j \in J}$ such that $\bigoplus_{j \in J}^{\mathrm{M}(E)} u_j = w$ and $u_j \leq x_j$ for all $j \in J$. This yields that $u_j = 0$ for all $j \in J$, i.e., $w = 0$. It follows that $\widehat{x} \wedge_B y = 0$. Applying the above considerations to $\widehat{x} \wedge_B y = 0$ we get that $\widehat{x} \wedge_B \widehat{y} = 0$.

Now, since $\widehat{x} \leftrightarrow_B \widehat{y}$ and $\widehat{x} \wedge_B \widehat{y}$ exists we have that $\widehat{x} \oplus (\widehat{y} \ominus (\widehat{x} \wedge_B \widehat{y})) = \widehat{x} \oplus \widehat{y}$ exists. It follows that $\widehat{x} \leq (\widehat{y})'$, i.e., $\widehat{x} \wedge \widehat{y} \leq (\widehat{y})' \wedge \widehat{y} = 0$. □

Theorem 3.16. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Then every block B in E is a lattice.*

Proof. Let B be a block and $y, z \in B$. Then by Corollary 3.11 $y = y_S \oplus y_M$, $z = z_S \oplus z_M$, $y_S, z_S \in B \cap \mathrm{S}(E)$, $y_M, z_M \in B \cap \mathrm{M}(E)$. Let us put $c = (y_S \wedge_B z_S) \oplus (y_S \wedge_B z_M) \oplus (z_S \wedge_B y_M) \oplus (y_M \wedge_B z_M) \in B$. Note that all the summands of c exist in virtue of Statement 1.10, (i) and Propositions 3.10, 3.14. Then clearly c is well defined since by Statement 1.10, (i) $(y_S \wedge_B z_S) \oplus (z_S \wedge_B y_M) = z_S \wedge_B (y_S \oplus y_M) \leq z_S$ and by Statement 1.10, (ii) $z_M = z_M \wedge_B (y_S \oplus y_S{}') = (z_M \wedge_B y_S) \oplus (z_M \wedge_B y_S{}') \geq (z_M \wedge_B y_S) \oplus (z_M \wedge_B y_M)$. Therefore also $c \leq z$ and by a symmetric argument we get that $c \leq y$. Hence $c \leq y, z$.

Let us show that $c = y \wedge_B z$. Assume now that $v \leq y, z$, $v \in B$.

By the Riesz decomposition property of B there are elements $v_1, v_2 \in B$ such that $v_1 \leq y_S$, $v_2 \leq y_M$ and $v = v_1 \oplus v_2 \leq z$. Again by the Riesz decomposition property we can find elements $v_{11}, v_{12}, v_{21}, v_{22} \in B$ such that $v_{11} \leq z_S$, $v_{12} \leq z_M$, $v_{21} \leq z_S$, $v_{22} \leq z_M$ and $v_1 = v_{11} \oplus v_{12}$, $v_2 = v_{21} \oplus v_{22}$. Hence $v = v_{11} \oplus v_{12} \oplus v_{21} \oplus v_{22} \leq (y_S \wedge_B z_S) \oplus (y_S \wedge_B z_M) \oplus (z_S \wedge_B y_M) \oplus (y_M \wedge_B z_M)$.

Consequently, c is the infimum of y, z. This yields that B is a lattice. □

Corollary 3.17. *Every block in a homogeneous orthocomplete effect algebra is an MV-algebra.*

Theorem 3.18. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Then E is Archimedean.*

Proof. In virtue of Proposition 2.7, it is sufficient to check that $\mathrm{M}(E)$ is Archimedean. Suppose $\mathrm{ord}(y) = \infty$. By Corollary 2.13, $ky \leq \widehat{y}$ for all $k \in \mathbb{N}$, and therefore $(k-1)y = ky \ominus y \leq \widehat{y} \ominus y \in \mathrm{M}(E)$ for all $k \in \mathbb{N} \subseteq \{0\}$. Hence there exists $\bigvee\{ky \mid y \in \mathbb{N}\}$ in $\mathrm{M}(E)$. By (IDL), y=0. □

Recall that a *Heyting* algebra (see [8]) is a system $(L, \leq, 0, 1, \wedge, \vee, \Rightarrow)$ consisting of a bounded lattice $(L, \leq, 0, 1, \wedge, \vee)$ and a binary operation $\Rightarrow : L \times L \to L$, called the *Heyting implication connective*, such that $x \wedge y \leq z$ iff $x \leq (y \Rightarrow z)$ for all $x, y, z \in L$. The *Heyting negation mapping* $^* : L \to L$ is defined by $x^* = (x \Rightarrow 0)$ for all $x \in L$. The set $L^* = \{x^* \mid x \in L\}$ is called the *Heyting center of L*.

A *pseudocomplementation* on a bounded lattice $(L, \leq, 0, 1, \wedge, \vee)$ is a mapping $^* : L \to L$ such that, for all $x, y \in L$, $x \wedge y = 0$ iff $x \leq y^*$. Then, for a Heyting algebra L, the Heyting negation is a pseudocomplementation on L.

Definition 3.19. [8] A *Heyting effect algebra* is a lattice effect algebra E that, as a bounded lattice, is also a Heyting algebra such that the Heyting center E^* coincides with the center $\mathrm{C}(E)$ of the effect algebra E.

Statement 3.20. [8, Theorem 5.2] Let E be a lattice effect algebra. Then the following conditions are equivalent:

(i) E is a Heyting effect algebra.
(ii) E is an MV-effect algebra with a pseudocomplementation $^* : E \to E$.

Theorem 3.21. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Then every block in E is an Archimedean Heyting effect algebra.*

Proof. Let B be a block of E. Then by Theorems 3.16 and 3.18 we have that B is an Archimedean MV-effect algebra. Let us define a pseudocomplementation $^* : B \to B$ on B. For any $x \in B$ we put $x^* = \widehat{x}' \in \mathrm{C}(B)$. Assume that $x, y \in B$. Then by Proposition 3.15 $x \wedge_B y = 0$ iff $\widehat{x} \wedge_B \widehat{y} = 0$ iff $\widehat{x} \le \widehat{y}'$ iff $\widehat{x} \le y^* \in \mathrm{S}(E)$ iff $x \le y^*$. Therefore B is an Archimedean Heyting effect algebra. □

Corollary 3.22. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Then E can be covered by Archimedean Heyting effect algebras.*

Proof. Every homogeneous effect algebra is covered by its blocks. □

Proposition 3.23. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra and let $x, y \in \mathrm{M}(E)$ and $v \in E$ such that x, y and v are in the same block B of E. Then*

(i) *We have that $v \wedge y$ exists and $v \wedge_B y = v \wedge y$.*
(ii) *If $\widehat{x} = \widehat{y}$ then $\widehat{x} \ominus (x \wedge y) \in \mathrm{M}(E)$.*
(iii) *If $\widehat{x} = \widehat{y}$ then $x \vee_{\mathrm{M}(E)} y$ exists and $x \vee_{\mathrm{M}(E)} y = x \vee_B y = x \vee_{[0,\widehat{x}]} y$.*
(iv) *If $\widehat{x} = \widehat{y}$ then $\widehat{x \wedge y} = \widehat{x}$.*
(v) *If $x \le v$ and $v = v_S \oplus v_M$ such that $v_S \in \mathrm{S}(E)$ and $v_M \in \mathrm{M}(E)$ then $x = (x \wedge v_S) \oplus (x \wedge v_M)$.*

Proof. According to Proposition 3.10, $\widehat{x}, \widehat{y} \in B$. By Proposition 3.14, $x \wedge y = x \wedge_{\mathrm{M}(E)} y$ exists and belongs to $\mathrm{M}(E)$. In virtue of Corollary 3.11, it belongs to B.
(i): Let $u \le v$ and $u \le y$. Since $u \in [0, y] \subseteq B$ we have that $u \le v \wedge_B y$.
(ii): Since $x, y \in \mathrm{M}(E)$ we have that $x \wedge y$ exists, $x \le \widehat{x}$, $y \le \widehat{x}$ and hence also $\widehat{x} \ominus (x \wedge y) \le \widehat{x}$ exists. Hence $(\widehat{x} \ominus (x \wedge y)) \oplus (x \wedge y) = \widehat{x}$. B contains $0, x, y, x \wedge y, \widehat{x} \ominus (x \wedge y), \widehat{x}$. Let us put $z = \widehat{x} \ominus (x \wedge y)$. Then $z = z_S \oplus z_M$, $z_S \in \mathrm{S}(E) \cap B = \mathrm{C}(B), z_S \le \widehat{x}$ and $z_M \in \mathrm{M}(E)$. Since z_S is central in B we have that $(\widehat{x} \ominus (x \wedge y)) \wedge_B z_S) \oplus ((x \wedge y) \wedge_B z_S) = z_S$. From (i) we get that $z_S \oplus (x \wedge (y \wedge z_S)) = z_S$ and $z_S \oplus (y \wedge (x \wedge z_S)) = z_S$. Hence $0 = x \wedge (y \wedge z_S) = y \wedge (x \wedge z_S)$. It follows from (i) that $0 = \widehat{x} \wedge (y \wedge z_S) = y \wedge z_S = \widehat{y} \wedge (x \wedge z_S) = x \wedge z_S$. We have that $z_S = z_S \wedge \widehat{x} = z_S \wedge_B (x \oplus (\widehat{x} \ominus x)) = (z_S \wedge_B x) \oplus (z_S \wedge_B (\widehat{x} \ominus x)) = z_S \wedge_B (\widehat{x} \ominus x) \le \widehat{x} \ominus x$. Since $\widehat{x} \ominus x \in \mathrm{M}(E)$ we obtain that $z_S = 0$ and $z_M = \widehat{x} \ominus (x \wedge y) \in \mathrm{M}(E)$.

(iii): Since $\widehat{x} \ominus x, \widehat{x} \ominus y \in \mathrm{M}(E)$ we have by (ii) that $\widehat{x} \ominus ((\widehat{x} \ominus x) \wedge (\widehat{x} \ominus y)) = x \vee_{[0,\widehat{x}]} y \in \mathrm{M}(E)$ exists. Let $z \geq x, y$, $z \in \mathrm{M}(E)$. Then $u = z \wedge (x \vee_{[0,\widehat{x}]} y) \in \mathrm{M}(E)$ exists, $u \geq x, y$, $u \leq x \vee_{[0,\widehat{x}]} y \leq \widehat{x}$. Hence $z \geq u \geq x \vee_{[0,\widehat{x}]} y$.
(iv): Clearly, $x \wedge y \leq \widehat{x \wedge y} \leq \widehat{x}$. This yields that $\widehat{x} \ominus \widehat{x \wedge y} \leq \widehat{x} \ominus x \wedge y \in \mathrm{M}(E)$, $\widehat{x} \ominus \widehat{x \wedge y} \in \mathrm{S}(E)$. It follows that $\widehat{x} \ominus \widehat{x \wedge y} = 0$, i.e., $\widehat{x \wedge y} = \widehat{x}$.
(v): Then $v_S \in \mathrm{C}(B)$ and $v_M \in B$. Hence $x = (x \wedge_B v_S) \oplus (x \wedge_B (v_S)')$, $x \wedge_B v_S, x \wedge_B (v_S)' \in B$. Moreover by (i) we have that $x \wedge_B v_S = x \wedge v_S$ and $x \wedge_B (v_S)' = x \wedge (v_S)'$. Evidently, $x \wedge (v_S)' \leq v = v_S \oplus v_M$. Since B has the Riesz decomposition property we have that $x \wedge (v_S)' = u_1 \oplus u_2$, $u_1 \leq v_S$ and $u_2 \leq v_M$. But $u_1 \leq x \wedge (v_S)'$ yields that $u_1 = 0$. Hence $x \wedge (v_S)' \leq v_M$. This yields that $x \wedge (v_S)' = x \wedge (v_S)' \wedge v_M = x \wedge v_M$. It follows that $x = (x \wedge v_S) \oplus (x \wedge v_M)$. ☐

The following theorem reminds us [3, Theorem 37] which was formulated for D-lattices.

Theorem 3.24. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra and let $x, y \in \mathrm{M}(E)$. Then the following conditions are equivalent:*

(i) $x \leftrightarrow y$.
(ii) $x \leftrightarrow_{\mathrm{M}(E)} y$.
(iii) $x \vee_{\mathrm{M}(E)} y$ *exists and* $(x \vee_{\mathrm{M}(E)} y) \ominus y = x \ominus (x \wedge y)$.

Proof. (i) $\Longrightarrow$ (ii): Since $x \leftrightarrow y$ there are $p, q, r \in E$ such that $x = p \oplus q$, $y = q \oplus r$ and $p \oplus q \oplus r$ exists. Clearly, $p, q \leq x$ and $q, r \leq y$. Hence $q \leq x \wedge y \in \mathrm{M}(E)$. Moreover, $x = (x \ominus (x \wedge y)) \oplus (x \wedge y)$, $y = (y \ominus (x \wedge y)) \oplus (x \wedge y)$ and $(x \ominus (x \wedge y)) \oplus (x \wedge y) \oplus (y \ominus (x \wedge y))$ exists since $p \oplus q \oplus r = x \oplus r$ exists and $y \ominus (x \wedge y) \leq r = y \ominus q$. Let us put $z = (x \ominus (x \wedge y)) \oplus (x \wedge y) \oplus (y \ominus (x \wedge y))$. Since E is sharply dominating we have that $z = z_S \oplus z_M$, $z_S \in \mathrm{S}(E)$ and $z_M \in \mathrm{M}(E)$.

Since $x \leftrightarrow y$ there is a block B of E such that $x, y, x \wedge y, z, z_S, z_M \in B$. Hence $z_S \in \mathrm{C}(B)$ and therefore $z_S = z_S \wedge ((x \ominus (x \wedge y)) \oplus (x \wedge y) \oplus (y \ominus (x \wedge y))) = ((z_S \wedge x) \ominus ((z_S \wedge x) \wedge (z_S \wedge y))) \oplus ((z_S \wedge x) \wedge (z_S \wedge y)) \oplus ((z_S \wedge y) \ominus ((z_S \wedge x) \wedge (z_S \wedge y)))$. Let us put $u = z_S \wedge x$ and $v = z_S \wedge y$. Then $z_S = (u \ominus (u \wedge v)) \oplus (u \wedge v) \oplus (v \ominus (u \wedge v))$, $u, v \in \mathrm{M}(E) \cap B$, $\widehat{u}, \widehat{v} \in B$.

Recall first that $\widehat{u} = \widehat{v} = \widehat{v \ominus (u \wedge v)} = z_S$. This follows immediately from Statement 2.8, (ii).

Since $(u \ominus (u \wedge v)) \wedge (v \ominus (u \wedge v)) = 0$ we have by [19, Proposition 3.4] that z_S is the minimal upper bound of u and v. Therefore $z_S = u \vee_{[0,z_S]} v$. From Proposition 3.23, (iii) we have that $u \vee_{[0,z_S]} v \in \mathrm{M}(E)$. Hence $z_S = 0$ and $z = z_M = (x \ominus (x \wedge y)) \oplus (x \wedge y) \oplus (y \ominus (x \wedge y)) \in \mathrm{M}(E)$, i.e., $x \leftrightarrow_{\mathrm{M}(E)} y$.
(ii) $\Longrightarrow$ (iii): Assume that $x \leftrightarrow_{\mathrm{M}(E)} y$, i.e., $z = (x \ominus (x \wedge y)) \oplus_{\mathrm{M}(E)} (x \wedge y) \oplus (y \ominus (x \wedge y))$ exists. Again by [19, Proposition 3.4] we have that $z \in \mathrm{M}(E)$ is the minimal upper bound of x and y. Let $m \in \mathrm{M}(E)$ be an upper bound of x and y. Then $m \wedge z$ is an upper bound of x and y, $m \wedge z \leq z$ and hence $m \geq m \wedge z = z$. It follows that $(x \vee_{\mathrm{M}(E)} y) \ominus y = x \ominus (x \wedge y)$.

(iii) $\Longrightarrow$ (i): It is enough to put $d = x \wedge y$ and $c = x \vee_{\mathrm{M}(E)} y$. Then $d \leq x \leq c$ and $d \leq y \leq c$ such that $c \ominus x = y \ominus d$, i.e., $x \leftrightarrow y$. $\square$

4. Triple Representation Theorem for orthocomplete homogeneous effect algebras

In what follows E will be always a homogeneous meager-orthocomplete sharply dominating effect algebra. Then $\mathrm{S}(E)$ is a sub-effect algebra of E and $\mathrm{M}(E)$ equipped with a partial operation $\oplus_{\mathrm{M}(E)}$ which is defined, for all $x, y \in \mathrm{M}(E)$, by $x \oplus_{\mathrm{M}(E)} y$ exists if and only if $x \oplus_E y$ exists and $x \oplus_E y \in \mathrm{M}(E)$ in which case $x \oplus_{\mathrm{M}(E)} y = x \oplus_E y$ is a generalized effect algebra. Moreover, we have a map $h : \mathrm{S}(E) \to 2^{\mathrm{M}(E)}$ that is given by $h(s) = \{x \in \mathrm{M}(E) \mid x \leq s\}$. As in [13] for complete lattice effect algebras we will prove the following theorem.

Triple Representation Theorem *The triple* $((\mathrm{S}(E), \oplus_{\mathrm{S}(E)}), (\mathrm{M}(E), \oplus_{\mathrm{M}(E)}), h)$ *characterizes* E *up to isomorphism within the class of all homogeneous meager-orthocomplete sharply dominating effect algebras.*

We have to construct an isomorphic copy of the original effect algebra E from the triple $(\mathrm{S}(E), \mathrm{M}(E), h)$. To do this we will first construct the following mappings in terms of the triple.

(M1) The mapping $\widehat{\ } : \mathrm{M}(E) \to \mathrm{S}(E)$.
(M2) For every $s \in \mathrm{S}(E)$, a partial mapping $\pi_s : \mathrm{M}(E) \to h(s)$, which is given by $\pi_s(x) = x \wedge_E s$ whenever $\pi_s(x)$ is defined.
(M3) The mapping $R : \mathrm{M}(E) \to \mathrm{M}(E)$ given by $R(x) = \widehat{x} \ominus_E x$.
(M4) The partial mapping $S : \mathrm{M}(E) \times \mathrm{M}(E) \to \mathrm{S}(E)$ given by $S(x, y)$ is defined if and only if the set $\mathcal{S}(x, y) = \{z \in \mathrm{S}(E) \mid z \wedge x$ and $z \wedge y$ exist, $z = (z \wedge x) \oplus_E (z \wedge y)\}$ has a top element $z_0 \in \mathcal{S}(x, y)$ in which case $S(x, y) = z_0$.

Since E is sharply dominating we have that, for all $x \in \mathrm{M}(E)$,

$$\widehat{x} = \bigwedge_E \{s \in \mathrm{S}(E) \mid x \in h(s)\} = \bigwedge_{\mathrm{S}(E)} \{s \in \mathrm{S}(E) \mid x \in h(s)\}.$$

Lemma 4.1. *Let* E *be a homogeneous meager-orthocomplete sharply dominating effect algebra,* $s \in \mathrm{S}(E)$ *and* $x \in \mathrm{M}(E)$*. Then*

(i) $\bigvee_{M(E)} \{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}$ *exists. Moreover, if* $x \wedge_E s$ *exists then* $x \wedge_E s \in \mathrm{M}(E)$ *and*

$$x \wedge_E s = \bigvee_E \{y \in E \mid y \leq x, y \leq s\} = \bigvee_{M(E)} \{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}.$$

(ii) *If* $x \leftrightarrow s$ *then* $x \wedge_E s$ *exists and*

$$x \wedge_E s = \bigvee_{M(E)} \{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}.$$

Proof. (i): Note that $\{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\} \subseteq [0, x]$. Since from Proposition 3.12 we have that $[0, x]$ is a complete MV-effect algebra we get that $z = \bigvee_{[0,x]}\{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}$ exists. Let us show that $z = \bigvee_{\mathrm{M}(E)}\{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}$. Let $m \in \mathrm{M}(E)$ such that m is an upper bound of the set $\{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}$. Since $m \wedge_E z \in \mathrm{M}(E)$ exists and $m \wedge_E z \in [0, x]$ is an upper bound of the set $\{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\}$ we have that $z \leq m \wedge_E z \leq m$.

Now, assume that $x \wedge_E s$ exists and let us check that $x \wedge_E s = z$. Clearly, $x \wedge_E s = \bigvee_E\{y \in E \mid y \leq x, y \leq s\} \in \mathrm{M}(E)$, i.e., $y \leq x \wedge_E s \leq z$ for all $y \in \mathrm{M}(E)$ such that $y \leq x$ and $y \leq s$. This yields $x \wedge_E s = z$.
(ii): By Statement 2.3, (e) there is some block B of E such that $x, s \in B$. Hence $s \in \mathrm{C}(B)$ by Statement 2.3, (g). This yields that $x \wedge_B s \in B$ exists and since $x \wedge_B s \leq x$ we have that $x \wedge_B s \in \mathrm{M}(E)$. ¿From Corollary 3.11 we know that $[0, x]_E \subseteq B$. Hence $x \wedge_B s \in \{y \in \mathrm{M}(E) \mid y \leq x, y \leq s\} \subseteq B$ and $z \in B$. This invokes that $x \wedge_B s \leq z$. Then $z \wedge_B s \in B$ exists in B, $z \wedge_B s \leq x$, $z \wedge_B s \leq s$ and, for all $y \in \mathrm{M}(E)$ such that $y \leq x$ and $y \leq s$, we have that $y \leq z \wedge_B s$. Hence $z \leq z \wedge_B s \leq s$. Altogether $z = x \wedge_B s$. Let us check that $z = x \wedge_E s$. Assume that $g \in E$, $g \leq s$ and $g \leq x$. Then $g \in B$ and therefore $g \leq x \wedge_B s$. It follows that $z = x \wedge_E s = x \wedge_B s$. □

Hence, for all $s \in \mathrm{S}(E)$ and for all $x \in \mathrm{M}(E)$, we put $z = \bigvee_{M(E)}\{y \in \mathrm{M}(E) \mid y \leq x, y \in h(s)\}$. Then $\pi_s(x)$ is defined if $z \in h(s)$ in which case

$$\begin{aligned}\pi_s(x) &= z = x \wedge_E s = \textstyle\bigvee_E\{y \in E \mid y \leq x, y \leq s\}\\ &= \textstyle\bigvee_E\{y \in \mathrm{M}(E) \mid y \leq x, y \in h(s)\} = \bigvee_{M(E)}\{y \in \mathrm{M}(E) \mid y \leq x, y \in h(s)\}.\end{aligned}$$

Now, let us construct the mapping R as in [13].

Lemma 4.2. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra and let $x \in \mathrm{M}(E)$. Then $y = \widehat{x} \ominus x$ is the only element such that*

(i) $y \in \mathrm{M}(E)$ *such that* $\widehat{y} = \widehat{x}$.
(ii) $x \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (x \wedge y))$ *exists and* $x \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (x \wedge y)) \in h(\widehat{x})$.
(iii) *For all* $z \in h(\widehat{x})$, $z \oplus_{\mathrm{M}(E)} x \in h(\widehat{x})$ *if and only if* $z \leq y$ *and* $\widehat{y \ominus_{\mathrm{M}(E)} z} = \widehat{x}$.

Proof. Let us prove that $y = \widehat{x} \ominus x$ satisfies (i)-(iii). By Statement 2.8 we get that (i) is satisfied. Evidently, $x \wedge y$ exists and $x \oplus y = \widehat{x}$. Invoking Theorem 3.24 we obtain that $x \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (x \wedge y)) \in \mathrm{M}(E)$. Let $z \in h(\widehat{x})$ and assume that $z \oplus_{\mathrm{M}(E)} x \in h(\widehat{x})$. Since $z \oplus_{\mathrm{M}(E)} x = z \oplus x \leq x \oplus y = \widehat{x}$ we get that $z \leq y$. Moreover, $(x \oplus z) \oplus (y \ominus z) = \widehat{x}$ yields again by Statement 2.8 that $\widehat{y \ominus z} = \widehat{x}$. Now, let $z \in h(\widehat{x})$, $z \leq y$ and $\widehat{y \ominus_{\mathrm{M}(E)} z} = \widehat{x}$. Then $\widehat{x} = x \oplus y = x \oplus (y \ominus z) \oplus z = (x \oplus z) \oplus (y \ominus z)$. Since $y \ominus z \in \mathrm{M}(E)$ we have that $x \oplus z \in \mathrm{M}(E)$. Hence $z \oplus_{\mathrm{M}(E)} x \in h(\widehat{x})$.

Let us verify that $y = \widehat{x} \ominus x$ is the only element satisfying (i)-(iii). Let the elements y_1 and y_2 of E satisfy (i)-(iii) and $y_1 \leftrightarrow y_2$. Let us put $u =$

$y_1 \wedge y_2$. Then $\widehat{u} = \widehat{x}$ by Proposition 3.23, (iv). By (ii) for y_1 we know that $x \vee_{\mathrm{M}(E)} y_1 \in \mathrm{M}(E)$ exists. Since $[0, x \vee_{\mathrm{M}(E)} y_1]$ is a complete lattice we have that $x \vee_{[0,x\vee_{\mathrm{M}(E)} y_1]} (y_1 \wedge y_2) \in \mathrm{M}(E)$ exists. Hence also $x \vee_{\mathrm{M}(E)} (y_1 \wedge y_2) \in \mathrm{M}(E)$ exists and $x \vee_{[0,x\vee_{\mathrm{M}(E)} y_1]} (y_1 \wedge y_2) = x \vee_{\mathrm{M}(E)} (y_1 \wedge y_2)$. This yields that $x \leftrightarrow (y_1 \wedge y_2)$ and from Theorem 3.24 we get that $x \oplus_{\mathrm{M}(E)} ((y_1 \wedge y_2) \ominus_{\mathrm{M}(E)} (x \wedge (y_1 \wedge y_2)))$ exists and $x \oplus_{\mathrm{M}(E)} ((y_1 \wedge y_2) \ominus_{\mathrm{M}(E)} (x \wedge (y_1 \wedge y_2))) = x \vee_{\mathrm{M}(E)} (y_1 \wedge y_2)$. Clearly, for any $z \in h(\widehat{x})$, we have $z \oplus_{\mathrm{M}(E)} x \in h(\widehat{x})$ iff $\left(z \leq y_1 \text{ and } \widehat{y_1 \ominus_{\mathrm{M}(E)} z} = \widehat{x}\right)$ and $\left(z \leq y_2 \text{ and } \widehat{y_2 \ominus_{\mathrm{M}(E)} z} = \widehat{x}\right)$ iff (by Proposition 3.23, (iv)) $\left(z \leq (y_1 \wedge y_2) \text{ and } \widehat{(y_1 \wedge y_2) \ominus_{\mathrm{M}(E)} z} = \widehat{x}\right)$. Hence $u = y_1 \wedge y_2$ satisfies (i)-(iii). Let us put $t = y_1 \ominus u$.

We will prove that, for all $n \in \mathbb{N}$, $(nt) \oplus_{\mathrm{M}(E)} x \in h(\widehat{x})$. For $n = 0$ the statement is true. Assume that the statement is valid for some $n \in \mathbb{N}$. Then by (iii) for u we have that $nt \leq u$ and $\widehat{u \ominus_{\mathrm{M}(E)} (nt)} = \widehat{x}$. Since $u \oplus t = y_1 \in \mathrm{M}(E)$ we get that $(n+1)t$ is defined, $(n+1)t \leq y_1 \leq \widehat{x}$, $u \ominus_{\mathrm{M}(E)} (nt) = y_1 \ominus_{\mathrm{M}(E)} ((n+1)t)$ and hence $\widehat{y_1 \ominus_{\mathrm{M}(E)} ((n+1)t)} = \widehat{x}$. By (iii) for y_1 we obtain that $((n+1)t) \oplus_{\mathrm{M}(E)} x \in h(\widehat{x})$. In particular, nt exists for all $n \in \mathbb{N}$. Since E is Archimedean, i.e., $t = 0$ and $y_1 \wedge y_2 = y_1$. This yields that $y_1 \leq y_2$. Interchanging y_1 with y_2 we get that $y_2 \leq y_1$, i.e., $y_1 = y_2$.

Now, let us assume that some y satisfies (i)-(iii) and put $y_1 = y$, $y_2 = \widehat{x} \ominus x$. Since $x \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (x \wedge y)) \leq \widehat{x}$ exists by (ii) we have that $x \leftrightarrow y$ and this yields that $\widehat{x} \leftrightarrow y$.

By [3, Theorem 36] we get that $y_1 = y \leftrightarrow (\widehat{x} \ominus x) = y_2$. Therefore $y = \widehat{x} \ominus x$. $\square$

What remains is the partial mapping S. Let $x, y \in \mathrm{M}(E)$. Note that by Statement 2.8, (ii) and Lemmas 4.1 and 4.2 $\mathcal{S}(x,y) = \{z \in \mathrm{S}(E) \mid z = (z \wedge x) \oplus_E (z \wedge y)\} = \{z \in \mathrm{S}(E) \mid \pi_z(x) \text{ and } \pi_z(x) \text{ are defined}, z = \widehat{\pi_z(x)} \text{ and } R(\pi_z(x)) = \pi_z(y)\}$. Hence whether $S(x,y)$ is defined or not we are able to decide in terms of the triple. Since the eventual top element z_0 of $\mathcal{S}(x,y)$ is in $\mathrm{S}(E)$ our definition of $S(x,y)$ is correct.

Lemma 4.3. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra, $x, y \in \mathrm{M}(E)$. Then $x \oplus_E y$ exists in E iff $S(x,y)$ is defined in terms of the triple $(\mathrm{S}(E), \mathrm{M}(E), h)$ and $(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)) \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y))$ exists in $\mathrm{M}(E)$ such that $(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)) \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)) \in h(S(x,y)')$. Moreover, in that case*

$$x \oplus_E y = \underbrace{S(x,y)}_{\in \mathrm{S}(E)} \oplus_E \underbrace{((x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)) \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)))}_{\in \mathrm{M}(E)}.$$

Proof. Assume first that $x \oplus_E y$ exists in E and let us put $z = x \oplus_E y$. Since E is sharply dominating we have that $z = z_S \oplus_E z_M$ such that $z_S \in \mathrm{S}(E)$ and $z_M \in \mathrm{M}(E)$. Since $x \leftrightarrow y$ by Statement 2.3, (e) there is a block B of E such

that $x, y, z \in B$. By Statement 2.11, (i) we obtain that $z_S, z_M \in B$. Therefore $z_S \in \mathrm{C}(B)$ and by Statement 1.10, (i) we have that $z_S = z_S \wedge (x \oplus_E y) = z_S \wedge (x \oplus_B y) = (z_S \wedge_B x) \oplus_B (z_S \wedge_B y) = (z_S \wedge x) \oplus_E (z_S \wedge y)$. Hence $z_S \in \mathcal{S}(x, y)$. Now, assume that $u \in \mathcal{S}(x, y)$. Then $u = (u \wedge x) \oplus_E (u \wedge y) \leq x \oplus_E y$. Since $u \in \mathrm{S}(E)$ we have that $u \leq z_S$, i.e., z_S is the top element of $\mathcal{S}(x, y)$. Moreover, we have

$$\begin{aligned} z_S \oplus_E z_M &= x \oplus_E y \\ &= \Big(\big(S(x,y) \wedge x\big) \oplus_E \big(x \ominus_E (S(x,y) \wedge x)\big) \Big) \oplus_E \\ &\qquad \Big(\big((S(x,y) \wedge y)\big) \oplus_E \big(y \ominus_E (S(x,y) \wedge y)\big) \Big) \\ &= S(x,y) \oplus_E \Big(\big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_E \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big) \Big). \end{aligned}$$

It follows that $z_M = \big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_E \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big)$, i.e., $z_M = \big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_{\mathrm{M}(E)} \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big)$ and evidently $z_M \in h(z_S')$.

Conversely, let us assume that $S(x, y)$ is defined in terms of $(\mathrm{S}(E), \mathrm{M}(E), h)$, $\big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_{\mathrm{M}(E)} \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big)$ exists in $\mathrm{M}(E)$ and $\big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_{\mathrm{M}(E)} \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big) \in h(S(x,y)')$. Then $\big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_{\mathrm{M}(E)} \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big) \leq S(x,y)'$, i.e.,

$$\begin{aligned} z &= S(x,y) \oplus_E \Big(\big(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)\big) \oplus_{\mathrm{M}(E)} \big(y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)\big) \Big) \\ &= \big((S(x,y) \wedge x) \oplus_E (S(x,y) \wedge y)\big) \oplus_E \\ &\quad \Big(\big(x \ominus_E (S(x,y) \wedge x)\big) \oplus_E \big(y \ominus_E (S(x,y) \wedge y)\big) \Big) = x \oplus_E y \end{aligned}$$

is defined. □

Theorem 4.4. *Let E be a homogeneous meager-orthocomplete sharply dominating effect algebra. Let $\mathrm{T}(E)$ be a subset of $\mathrm{S}(E) \times \mathrm{M}(E)$ given by*

$$\mathrm{T}(E) = \{(z_S, z_M) \in \mathrm{S}(E) \times \mathrm{M}(E) \mid z_M \in h(z_S')\}.$$

Equip $\mathrm{T}(E)$ with a partial binary operation $\oplus_{\mathrm{T}(E)}$ with $(x_S, x_M) \oplus_{\mathrm{T}(E)} (y_S, y_M)$ is defined if and only if

(i) *$S(x_M, y_M)$ is defined,*
(ii) *$z_S = x_S \oplus_{\mathrm{S}(E)} y_S \oplus_{\mathrm{S}(E)} S(x_M, y_M)$ is defined,*
(iii) *$z_M = \big(x_M \ominus_{\mathrm{M}(E)} (S(x_M, y_M) \wedge x_M)\big) \oplus_{\mathrm{M}(E)} \big(y_M \ominus_{\mathrm{M}(E)} (S(x_M, y_M) \wedge y_M)\big)$ is defined,*
(iv) *$z_M \in h(z_S')$.*

In this case $(z_S, z_M) = (x_S, x_M) \oplus_{\mathrm{T}(E)} (y_S, y_M)$. Let $0_{\mathrm{T}(E)} = (0_E, 0_E)$ and $1_{\mathrm{T}(E)} = (1_E, 0_E)$. Then $\mathrm{T}(E) = (\mathrm{T}(E), \oplus_{\mathrm{T}(E)}, 0_{\mathrm{T}(E)}, 1_{\mathrm{T}(E)})$ is an effect algebra and the mapping $\varphi : E \to \mathrm{T}(E)$ given by $\varphi(x) = (\widetilde{x}, x \ominus_E \widetilde{x})$ is an isomorphism of effect algebras.

Proof. Evidently, φ is correctly defined since, for any $x \in E$, we have that $x = \widetilde{x} \oplus_E (x \ominus \widetilde{x}) = x_S \oplus_E x_M$, $x_S \in \mathrm{S}(E)$ and $x_M \in \mathrm{M}(E)$. Hence $\varphi(x) = (x_S, x_M) \in \mathrm{S}(E) \times \mathrm{M}(E)$ and $x_M \in h(x'_S)$. Let us check that φ is bijective. Assume first that $x, y \in E$ such that $\varphi(x) = \varphi(y)$. We have $x = \widetilde{x} \oplus_E (x \ominus_E \widetilde{x}) = \widetilde{y} \oplus_E (y \ominus_E \widetilde{y}) = y$. Hence φ is injective. Let $(x_S, x_M) \in \mathrm{S}(E) \times \mathrm{M}(E)$ and $x_M \in h(x'_S)$. This yields that $x = x_S \oplus_E x_M$ exists and evidently by Lemma 2.9, (i) $\widetilde{x} = x_S$ and $x \ominus_E \widetilde{x} = x_M$. It follows that φ is surjective. Moreover, $\varphi(0_E) = (0_E, 0_E) = 0_{\mathrm{T}(E)}$ and $\varphi(1_E) = (1_E, 0_E) = 1_{\mathrm{T}(E)}$.

Now, let us check that, for all $x, y \in E$, $x \oplus_E y$ is defined iff $\varphi(x) \oplus_{\mathrm{T}(E)} \varphi(y)$ is defined in which case $\varphi(x \oplus_E y) = \varphi(x) \oplus_{\mathrm{T}(E)} \varphi(y)$. For any $x, y, z \in E$ we obtain

$$
\begin{aligned}
& z = x \oplus_E y \iff \\
& z = \big(\widetilde{x} \oplus_E (x \ominus_E \widetilde{x})\big) \oplus_E \big(\widetilde{y} \oplus_E (y \ominus_E \widetilde{y})\big) \iff \\
& z = (\widetilde{x} \oplus_E \widetilde{y}) \oplus_E \big((x \ominus_E \widetilde{x}) \oplus_E (y \ominus_E \widetilde{y})\big) \iff \\
& \text{by Lemma 4.3 } (\exists u \in E)\ u = S(x \ominus_E \widetilde{x}, y \ominus_E \widetilde{y}) \text{ and} \\
& z = (\widetilde{x} \oplus_E \widetilde{y}) \oplus_E \Big(u \oplus_E \big((x \ominus_E \widetilde{x}) \ominus_E (u \wedge (x \ominus_E \widetilde{x}))\big) \\
& \qquad\qquad \oplus_E \big((y \ominus_E \widetilde{y}) \ominus_E (u \wedge (y \ominus_E \widetilde{y}))\big)\Big) \\
& \iff (\exists u \in E)\ u = S(x \ominus_E \widetilde{x}, y \ominus_E \widetilde{y}) \text{ and} \\
& z = (\widetilde{x} \oplus_E \widetilde{y} \oplus_E u) \oplus_E \Big(\big((x \ominus_E \widetilde{x}) \ominus_E (u \wedge (x \ominus_E \widetilde{x}))\big) \\
& \qquad\qquad \oplus_E \big((y \ominus_E \widetilde{y}) \ominus_E (u \wedge (y \ominus_E \widetilde{y}))\big)\Big) \\
& \iff (\exists u \in E)\ u = S(x \ominus_E \widetilde{x}, y \ominus_E \widetilde{y}) \text{ and} \\
& z = (\widetilde{x} \oplus_{\mathrm{S}(E)} \widetilde{y} \oplus_{\mathrm{S}(E)} u) \\
& \qquad\qquad \oplus_E \Big(\big((x \ominus_E \widetilde{x}) \ominus_{\mathrm{M}(E)} (u \wedge (x \ominus_E \widetilde{x}))\big) \\
& \qquad\qquad \oplus_{\mathrm{M}(E)} \big((y \ominus_E \widetilde{y}) \ominus_{\mathrm{M}(E)} (u \wedge (y \ominus_E \widetilde{y}))\big)\Big) \\
& \iff (\widetilde{x}, x \ominus_E \widetilde{x}) \oplus_{\mathrm{T}(E)} (\widetilde{y}, y \ominus_E \widetilde{y}) \text{ is defined and} \\
& \varphi(z) = \Big(\widetilde{x} \oplus_{\mathrm{S}(E)} \widetilde{y} \oplus_{\mathrm{S}(E)} S(x \ominus_E \widetilde{x}, y \ominus_E \widetilde{y}), \big((x \ominus_E \widetilde{x}) \ominus \\
& \quad (S(x \ominus_E \widetilde{x}, y \ominus_E \widetilde{y}) \wedge (x \ominus_E \widetilde{x}))\big) \oplus_{\mathrm{M}(E)} \big((y \ominus_E \widetilde{y}) \\
& \quad \ominus (S(x \ominus_E \widetilde{x}, y \ominus_E \widetilde{y}) \wedge (y \ominus_E \widetilde{y}))\big)\Big) \\
& \quad = (\widetilde{x}, x \ominus_E \widetilde{x}) \oplus_{\mathrm{T}(E)} (\widetilde{y}, y \ominus_E \widetilde{y}) = \varphi(x) \oplus_{\mathrm{T}(E)} \varphi(y).
\end{aligned}
$$

Altogether, $\mathrm{T}(E) = (\mathrm{T}(E), \oplus_{\mathrm{T}(E)}, 0_{\mathrm{T}(E)}, 1_{\mathrm{T}(E)})$ is an effect algebra and the mapping $\varphi : E \to \mathrm{T}(E)$ is an isomorphism of effect algebras. $\square$

The Triple Representation Theorem then follows immediately.

Remark 4.5. Recall that our method may be also used in the case of complete lattice effect algebras as a substitute of the method from [13]. Moreover, since any homogeneous orthocomplete effect algebra E is both meager-orthocomplete and sharply dominating the Triple Representation Theorem is valid within the class of homogeneous orthocomplete effect algebras which was an open question asked by Jenča in [13].

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